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Generalized t–j model*

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Abstract. By parametrizing the t–j model we present a new electron correlation model with one free parameter. In one dimension, this model is of $SU_q(1|2)$ symmetry. The energy spectra are shown to be modulated by the free parameter in the model. The solution and symmetric structures of the Hilbert space, as well as the Bethe ansatz approach, are discussed for special cases.

Strongly correlated electronic systems are believed to be important in studying the phenomenon of high-temperature superconductivity [1, 2]. An appropriate starting model suggested by Anderson is the t–j model [3, 4]. The model describes the behaviour of electrons on a discrete lattice with a Hamiltonian including nearest-neighbour hopping (t) and antiferromagnetic exchange (j). Also, the Hilbert space admits no double occupancy of any single site. As the two-dimensional systems may share features of one-dimensional systems [5], the t–j model in one dimension has been investigated extensively. The model is shown to be integrable and supersymmetric when $j = \pm 2t$ [6, 7]. Nevertheless, the phenomenon of high-temperature superconductivity depends greatly on detailed material. While in the supersymmetric t–j model there are no non-trivial free parameters left. By taking into account some physical considerations, we present in this paper a generalized t–j model. The main point of interest is that this model degenerates into an integrable one with a free parameter q and q-deformed supersymmetric symmetry.

Electrons on a lattice are described by canonical Fermi operators $c_{j\sigma}^{+}$ and $c_{j\sigma}$ satisfying anti-commutation relations given by

$$\{c_{i\sigma}^{+}, c_{j\sigma'}\} = \delta_{ij}\delta_{\sigma\sigma'} \tag{1}$$

where $\sigma = \uparrow, \downarrow$; $j = 1, \ldots, L$ and L is the total number of lattice sites. $c_{i\sigma}$ annihilates an electron of spin σ at site i. The Fock vacuum $|0\rangle$ satisfies $c_{i\sigma}|0\rangle = 0$. As double occupancy is not allowed, there are three possible electronic states at a given lattice site i:

$$|0\rangle \qquad |+\rangle \equiv c_{j\uparrow}^{+}|0\rangle = |\uparrow\rangle \qquad |-\rangle \equiv c_{j\downarrow}^{+}|0\rangle = |\downarrow\rangle\,.$$

The Hamiltonian of the supersymmetric t–j model on a lattice of L sites is given by the following expression:

$$H_{tj} = \sum_{i=1}^{L-1}\left[X_i^{+-}X_{i+1}^{-+} + X_i^{-+}X_{i+1}^{+-} - X_i^{-0}X_{i+1}^{0-} + X_i^{0-}X_{i+1}^{-0} - X_i^{+0}X_{i+1}^{0+} + X_i^{0+}X_{i+1}^{+0} + n_i^{+}n_{i+1}^{+} + n_i^{-}n_{i+1}^{-} - n_i^{0}n_{i+1}^{0}\right] \tag{2}$$

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where

$$X_i^{\alpha\beta} = |\alpha\rangle_{i\,i}\langle\beta| \qquad \alpha,\ \beta = 0, +, - \tag{3}$$

are the local generators of the supersymmetric algebra $SU(1|2)$ and

$$n_i^0 = |0\rangle_{i\,i}\langle 0| \qquad n_i^+ = |+\rangle_{i\,i}\langle +| \qquad n_i^- = |-\rangle_{i\,i}\langle -| \tag{4}$$

are the number operators of holes, spin-up electrons and spin-down electrons at site i, respectively. It can be proved directly that H_{tj} commutes with the total operators of $SU(1|2)$ on the lattice.

In parametrizing the usual supersymmetric t–j Hamiltonian (2), we can reasonably distinguish the interactions between spin-up (-down) electrons and between holes. Therefore the coupling constants of $n_i^+ n_{i+1}^+$ and $n_i^- n_{i+1}^-$ are different to that of $n_i^0 n_{i+1}^0$.

We also suppose that the chemical potentials of spin-up (-down) electrons and holes are different. The terms for chemical potential are then of the form

$$\mu = \sum_{i=1}^{L} c\, n_i^0 + c'\,(n_i^+ + n_i^-) \tag{5}$$

where c and c' are chemical potentials of a hole and an electron, respectively. Accounting for the no-double-occupancy condition,

$$n_i^0 + n_i^+ + n_i^- = 1 \tag{6}$$

the chemical potential simply becomes

$$\mu = \sum_{i}^{L} n_i^0\,(c - c') \tag{7}$$

modular a constant term.

In addition, we consider the nearest-neighbour interactions between electrons with opposed spin direction. For an electron $c_{j\sigma}^+$ located at site j, the general form of the interaction is $n_{j-1}^{\bar\sigma} n_j^{\sigma} + n_j^{\sigma} n_{j+1}^{\bar\sigma}$, where σ takes $\uparrow$ and $\bar\sigma$ takes $\downarrow$ and vice versa. If we impose the saturation condition on it, the interaction only contributes one term to the Hamiltonian. Without loss of generality, we can assume that the interaction between opposed direction spin electrons is

$$\sum_{i=1}^{L-1} n_i^- n_{i+1}^+ \tag{8}$$

which may be, in a sense, considered as the first-order expansion of the Cooper pair in coordinate space. In the following, we will see that this term plays an important role in resumption of the symmetry of the system.

Based on the above analysis and accounting for suitable boundary conditions, we present a modified t–j Hamiltonian with parameters

$$H = \sum_{i=1}^{L-1} \big[X_i^{+-} X_{i+1}^{-+} + X_i^{-+} X_{i+1}^{+-} - X_i^{-0} X_{i+1}^{0-} + X_i^{0-} X_{i+1}^{-0} - X_i^{+0} X_{i+1}^{0+} + X_i^{0+} X_{i+1}^{+0}$$
$$+\lambda n_i^- n_{i+1}^+ + \gamma (n_i^+ n_{i+1}^+ + n_i^- n_{i+1}^-) + \theta n_i^0 n_{i+1}^0 + \tau n_i^0\big] \tag{9}$$

where $\lambda, \gamma, \theta, \tau$ are free parameters. This Hamiltonian no longer possesses the $SU(1|2)$ symmetry and is not integrable in general. To recover the broken symmetry in some contents we note that the generators $X_i^{\alpha\beta}$, $\alpha,\ \beta = 0, +, -$ in (3) are also the trivial representations of the q-deformed algebra $SU_q(1|2)$. Therefore a reasonable candidate for

symmetry imposed on the system is $SU_q(2)$. Before studying the symmetry of the system, we give some preliminary knowledge of $SU_q(1|2)$. This algebra is spanned by generators $X^{\alpha\beta}$, $\alpha, \beta = +, -, 0$. They satisfy the following algebraic relations:

$$\begin{aligned}
&[X^{+-}, X^{-+}] = [X^{++} - X^{--}]_q \qquad [X^{-0}, X^{0-}] = [X^{--} + X^{00}]_q \\
&[X^{++}, X^{+-}] = X^{+-} \qquad [X^{++}, X^{+0}] = X^{+0} \\
&[X^{++}, X^{-+}] = -X^{-+} \qquad [X^{++}, X^{0+}] = -X^{0+} \\
&[X^{--}, X^{-+}] = X^{-+} \qquad [X^{--}, X^{-0}] = X^{-0} \\
&[X^{--}, X^{+-}] = -X^{+-} \qquad [X^{--}, X^{0-}] = -X^{0-} \\
&[X^{00}, X^{0+}] = X^{0+} \qquad [X^{00}, X^{0-}] = X^{0-} \\
&[X^{00}, X^{+0}] = -X^{+0} \qquad [X^{00}, X^{-0}] = -X^{-0} \\
&(X^{-0})^2 = 0 \qquad (X^{0-})^2 = 0
\end{aligned} \tag{10}$$

and the Sierre relations

$$\begin{aligned}
&(X^{+-})^2 X^{-0} - (q + q^{-1}) X^{+-} X^{-0} X^{+-} + X^{-0} (X^{+-})^2 = 0 \\
&(X^{-+})^2 X^{0-} - (q + q^{-1}) X^{-+} X^{0-} X^{-+} + X^{0-} (X^{-+})^2 = 0
\end{aligned} \tag{11}$$

where

$$[X]_q \equiv \frac{q^x - q^{-x}}{q - q^{-1}}.$$

As a quantum algebra, $SU_q(1|2)$ has non-trivial Hopf algebraic structures with operations co-product, co-unit and antipode. Here we only give the co-product expressions:

$$\Delta(X^{\alpha\alpha}) = X^{\alpha\alpha} \otimes \mathbf{1} + \mathbf{1} \otimes X^{\alpha\alpha} \qquad \alpha = +, -, 0$$

$$\begin{aligned}
\Delta(X^{-+}) &= X^{-+} \otimes q^{(X^{++} - X^{--})/2} + q^{-(X^{++} - X^{--})/2} \otimes X^{-+} \\
&= X^{-+} \otimes \{1 + (q^{\frac{1}{2}} - 1) X^{++} + (q^{-\frac{1}{2}} - 1) X^{--}\} \\
&\quad + \{1 + (q^{-\frac{1}{2}} - 1) X^{++} + (q^{\frac{1}{2}} - 1) X^{--}\} \otimes X^{-+}
\end{aligned}$$

$$\begin{aligned}
\Delta(X^{+-}) &= X^{+-} \otimes q^{(X^{++} - X^{--})/2} + q^{-(X^{++} - X^{--})/2} \otimes X^{+-} \\
&= X^{+-} \otimes \{1 + (q^{\frac{1}{2}} - 1) X^{++} + (q^{-\frac{1}{2}} - 1) X^{--}\} \\
&\quad + \{1 + (q^{-\frac{1}{2}} - 1) X^{++} + (q^{\frac{1}{2}} - 1) X^{--}\} \otimes X^{+-}
\end{aligned}$$

$$\begin{aligned}
\Delta(X^{0-}) &= X^{0-} \otimes q^{(X^{00} + X^{--})/2} + q^{-(X^{00} + X^{--})/2} \otimes X^{0-} \\
&= X^{0-} \otimes \{1 + (q^{\frac{1}{2}} - 1)(X^{--} + X^{00})\} \\
&\quad + \{1 + (q^{-\frac{1}{2}} - 1)(X^{--} + X^{00})\} \otimes X^{0-}
\end{aligned}$$

$$\begin{aligned}
\Delta(X^{-0}) &= X^{-0} \otimes q^{(X^{00} + X^{--})/2} + q^{-(X^{00} + X^{--})/2} \otimes X^{-0} \\
&= X^{-0} \otimes \{1 + (q^{\frac{1}{2}} - 1)(X^{--} + X^{00})\} \\
&\quad + \{1 + (q^{-\frac{1}{2}} - 1)(X^{--} + X^{00})\} \otimes X^{-0}
\end{aligned} \tag{12}$$

$$\begin{aligned}
\Delta(X^{0+}) &= X^{0+} \otimes q^{(X^{++} + X^{00})/2} + q^{-(X^{++} + X^{00})/2} \otimes X^{-+} \\
&\quad + (q - q^{-1}) q^{-(X^{++} - X^{--})/2} X^{0-} \otimes q^{(X^{--} + X^{00})/2} X^{-+} \\
&= X^{0+} \otimes \{1 + (q^{\frac{1}{2}} - 1)(X^{++} + X^{00})\} \\
&\quad + \{1 + (q^{-\frac{1}{2}} - 1)(X^{++} + X^{00})\} \otimes X^{0+} + (q - q^{-1}) X^{0-} \otimes X^{-+}
\end{aligned}$$

$$\begin{aligned}\Delta(X^{+0}) &= X^{+0}\otimes q^{(X^{++}+X^{00})/2}+q^{-(X^{++}+X^{00})/2}\otimes X^{+0}\\ &\quad +(q-q^{-1})q^{-(X^{++}-X^{--})/2}X^{-0}\otimes q^{(X^{--}+X^{00})/2}X^{+-}\\ &= X^{+0}\otimes\{1+(q^{\frac{1}{2}}-1)(X^{++}+X^{00})\}\\ &\quad +\{1+(q^{-\frac{1}{2}}-1)(X^{++}+X^{00})\}\otimes X^{+0}+(q-q^{-1})X^{-0}\otimes X^{+-}.\end{aligned}$$

The co-product operator Δ are algebraic isomorphism, $\Delta(ab)=\Delta(a)\Delta(b)$, $\forall a,b\in U_q(1|2)$. X^{0+} and X^{+0} are the algebra elements in the sense that

$$X^{+0}=q^{\frac{1}{2}}X^{+-}X^{-0}-q^{-\frac{1}{2}}X^{-0}X^{-+}\qquad X^{0+}=q^{-\frac{1}{2}}X^{0-}X^{-+}-q^{\frac{1}{2}}X^{-+}X^{0-}. \tag{13}$$

Their co-product representations $\Delta(X^{+0})$ and $\Delta(X^{0+})$ are obtained by the actions of the co-product operator Δ. It is not difficult to show that operators defined by (12) satisfy the same relations (10) as $X^{\alpha\beta}$.

Now we return to study what conditions will be imposed on the parameters appearing in the Hamiltonian (9), when the system enjoys the symmetry of quantum group $SU_q(1|2)$. Define

$$\mathcal{X}^{\alpha\beta}=\Delta^{L-1}(X^{\alpha\beta})=(\Delta\otimes id)\Delta^{L-2}(X^{\alpha\beta}). \tag{14}$$

Explicitly

$$\begin{aligned}\mathcal{X}^{\alpha\alpha}&=\sum_{i=1}^{L}\mathbf{1}\otimes\cdots\otimes\mathbf{1}\otimes X^{\alpha\alpha}\otimes\mathbf{1}\otimes\cdots\otimes\mathbf{1}\qquad \alpha=+,-,0\\ \mathcal{X}^{\bar\alpha\alpha}&=\sum_{i=1}^{L}q^{-H_1/2}\otimes\cdots\otimes q^{-H_1/2}\otimes X^{\bar\alpha\alpha}\otimes q^{H_1/2}\cdots\otimes q^{H_1/2}\qquad \alpha,\bar\alpha=+,-\\ \mathcal{X}^{0-}&=\sum_{i=1}^{L}q^{-H_2/2}\otimes\cdots\otimes q^{-H_2/2}\otimes X^{0-}\otimes q^{H_2/2}\cdots\otimes q^{H_2/2}\\ \mathcal{X}^{-0}&=\sum_{i=1}^{L}q^{-H_2/2}\otimes\cdots\otimes q^{-H_2/2}\otimes X^{-0}\otimes q^{H_2/2}\cdots\otimes q^{H_2/2}\end{aligned} \tag{15}$$

where $H_1=X^{++}-X^{--}$ and $H_2=X^{00}+X^{--}$. The operator $\mathcal{X}^{0+}$ ($\mathcal{X}^{+0}$) can also be represented in terms of X^{-+} and X^{0-} (X^{+-} and X^{-0}). These operators satisfy the $SU_q(1|2)$ algebraic relations (10).

It is easy to show that the bosonic operators $\mathcal{X}^{\alpha\alpha}$ commute with the Hamiltonian (9) for arbitrary parameters λ, γ, θ and τ. Therefore we only need to calculate

$$\begin{aligned}[\mathcal{X}^{+-},H]&=\sum_{j=1}^{L-1}q^{-H_1/2}\otimes\cdots\otimes q^{-H_1/2}\otimes[q^{-H_1/2}\otimes X^{+-}_{j+1},H_{j\,j+1}]\otimes q^{H_1/2}\otimes\cdots\otimes q^{H_1/2}\\ &\quad +q^{-H_1/2}\otimes\cdots\otimes q^{-H_1/2}\otimes[X^{+-}_{j}\otimes q^{H_1/2},H_{j\,j+1}]\otimes q^{H_1/2}\otimes\cdots\otimes q^{H_1/2}\\ &=\sum_{j=1}^{L-1}(q^{1/2}-q^{-1/2}\gamma)X^{++}_{j}X^{+-}_{j+1}+(q^{1/2}\lambda-q^{-1/2}+q^{1/2}\gamma)X^{--}_{j}X^{+-}_{j+1}\\ &\quad +(q^{-1/2}-q^{1/2}\gamma+q^{1/2}\lambda)X^{+-}_{j}X^{++}_{j+1}+(-q^{1/2}+q^{-1/2}\gamma)X^{+-}_{j}X^{--}_{j+1}.\end{aligned} \tag{16}$$

Then the following conditions are necessary for $[\mathcal{X}^{+-},H]=0$:

$$\gamma=q\qquad \lambda=q-q^{-1}. \tag{17}$$

Similarly, the vanishing condition for commutator $[\mathcal{X}^{0+}, H] = 0$ imposes the conditions $\tau = \lambda$ and $\theta = -\gamma$. Taking $\lambda = \tau = q - q^{-1}$ and $\gamma = -\theta = q$ in (9) we get a parametrized Hamiltonian with q-deformed symmetry $SU_q(1|2)$,

$$H_{tj}^q = \sum_{i=1}^{L-1} \left[X_i^{+-} X_{i+1}^{-+} + X_i^{-+} X_{i+1}^{+-} - X_i^{-0} X_{i+1}^{0-} + X_i^{0-} X_{i+1}^{-0} - X_i^{+0} X_{i+1}^{0+} + X_i^{0+} X_{i+1}^{+0} \right.$$
$$\left. + q(n_i^+ n_{i+1}^+ + n_i^- n_{i+1}^- - n_i^0 n_{i+1}^0) + (q - q^{-1})(n_i^- n_{i+1}^+ + n_i^0) \right] . \tag{18}$$

The calculations to check that the other operators commutes with H_{tj}^q are tedious but direct. Therefore the modified t–j Hamiltonian (18) is of $SU_q(1|2)$ symmetry. It has a free parameter q. In order to investigate the meaning of parameter q, we rewrite the Hamiltonian in term of creation and annihilation operators of electrons:

$$H_{tj}^q = \sum_{i=1}^{L-1} \sum_{\sigma} \left[c_{i\sigma}^+ c_{i+1\sigma}^+ + \mathrm{HC} \right] + \frac{q - q^{-1}}{2} \sum_{i=1}^{L-1} \left[n_i s_{i+1}^z - s_i^z n_{i+1} \right]$$
$$+ 2 \sum_{i=1}^{L-1} \left[s_i^x s_{i+1}^x + s_i^y s_{i+1}^y + \frac{q + q^{-1}}{2} \left(s_i^z s_{i+1}^z - \tfrac{1}{4} \right) \delta_{n_i,1} \delta_{n_{i+1},1} \right]$$
$$+ (q + q^{-1}) \sum_{i=1}^{L-1} n_i + q(n_L - n_1) - q^{-1}(L - 1) . \tag{19}$$

It is clear that q describes the anisotropic electron spin interaction and the electron chemical potential. But the q-dependence of electron chemical potential is superficial. This stems from the fact that the eigenstates of H_{tj}^q are also the eigenstates of $H_{tj}^q + \mu \sum n_i$, but only with different eigenvalues. So q reveals only the spin interaction. We point out here that the new system is $SU_q(1|2)$ symmetry broken. Now we consider the classical limit of H_{tj}^q ($q \to 1$). It is easy to show the Hamiltonian degenerates into the usual supersymmetric t–j model ($t = 1$, $j = 2$). Besides, when the system satisfies the half-filling condition ($n_i = 1$), we can get a submodel which enjoys $SU_q(2)$ symmetry—H_{xxz} ferromagnetic model with fixed boundary terms.

This generalized t–j model is again integrable. Similarly to the case of supersymmetric t–j model [6], it can be exactly solved in terms of the algebraic Bethe ansatz. The symmetry algebra is now $SU_q(1|2)$. Moreover, the usual coordinate Bethe ansatz also gives complete energy spectra and eigenstates of the generalized model by using the symmetry algebra operators. The detailed exact solutions and the phase diagram analysis are to appear in another paper [10]. In this paper we calculate the model at a few lattice sites, so as to show the symmetric structures of the Hilbert space and the roles played by the free parameter q.

First we consider the case $L = 2$. The Hamiltonian is simply

$$H = X^{+-} \otimes X^{-+} + X^{-+} \otimes X^{+-} - X^{-0} \otimes X^{0-} + X^{0-} \otimes X^{-0} - X^{+0} \otimes X^{0+} + X^{0+} \otimes X^{+0}$$
$$+ q(n^+ \otimes n^+ + n^- \otimes n^- - n^0 \otimes n^0) + (q - q^{-1})(n^- \otimes n^+ + n^0 \otimes 1) . \tag{20}$$

By $|\alpha\beta\rangle$ we denote $|\alpha\rangle \otimes |\beta\rangle$ for $\alpha, \beta = +, -, 0$. The ferromagnetic states $\Psi_{\uparrow\uparrow} = |\uparrow\uparrow\rangle$ and $\Psi_{\downarrow\downarrow} = |\downarrow\downarrow\rangle$ are eigenstates of H with energy q. The two-hole state $\Psi_{00} = |00\rangle$ is also an eigenstate with energy $-q^{-1}$. For the configuration of one spin-up electron and one spin-down electron, the state is of the form $\Psi_{\uparrow\downarrow} = a|\uparrow\downarrow\rangle + b|\downarrow\uparrow\rangle$, where a and b are constants. The Schrödinger equation

$$H\Psi_{\uparrow\downarrow} = E\Psi \tag{21}$$

gives two solutions of E, $E_1 = q$, $E_2 = -q^{-1}$ with eigenstates

$$\begin{aligned} \Psi^1_{\uparrow\downarrow} &= q|\downarrow\uparrow\rangle + |\uparrow\downarrow\rangle \\ \Psi^2_{\uparrow\downarrow} &= |\downarrow\uparrow\rangle - q|\uparrow\downarrow\rangle \end{aligned} \tag{22}$$

respectively.

For the configuration with a hole and a spin-up electron we have

$$\begin{aligned} \Psi^1_{0\uparrow} &= q|0\uparrow\rangle + |\uparrow 0\rangle \\ \Psi^2_{0\uparrow} &= |0\uparrow\rangle - q|\uparrow 0\rangle \end{aligned} \tag{23}$$

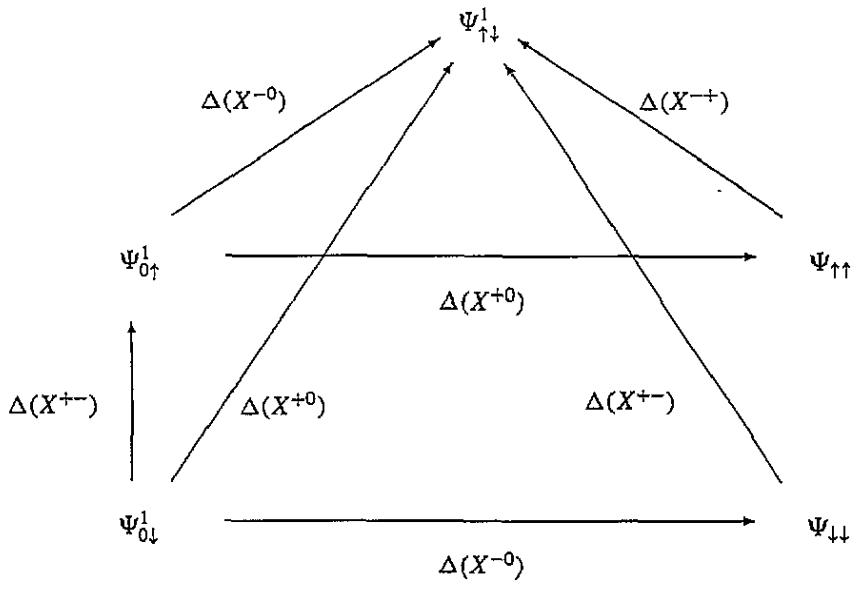
with energy E_1 and E_2, respectively.

The one spin-down electron and one-hole case also gives two eigenstates

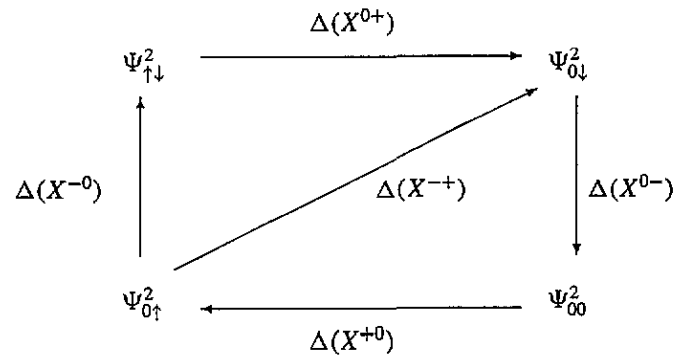
$$\begin{aligned} \Psi^1_{0\downarrow} &= q|0\downarrow\rangle + |\downarrow 0\rangle \\ \Psi^2_{0\downarrow} &= |0\downarrow\rangle - q|\downarrow 0\rangle \end{aligned} \tag{24}$$

with respect to energy E_1 and E_2.

Therefore there are 9 ($= 3^L = 3^2$) independent states. According to the tensor decomposition of the $SU_q(1|2)$ algebra representation space, there are two invariant spaces. The eigenstates in the same invariant space can be exchanged by using the co-product operators of the algebra. One of the invariant subspaces is constituted of states $\Psi_{\uparrow\uparrow}$, $\Psi^1_{\uparrow\downarrow}$, $\Psi^1_{0\uparrow}$, $\Psi^1_{0\downarrow}$ and $\Psi_{\downarrow\downarrow}$ with energy q. They satisfy the following exchange diagram:



The left four states give rise to another invariant eigenstate space of energy $-q^{-1}$,



where the inverse action of $\Delta(X^{\alpha\beta})$ is $\Delta(X^{\beta\alpha})$ for $\alpha \neq \beta = +, -, 0$.

For $L = 3$ there are 27 independent eigenstates. They decompose to four invariant subspaces of the algebra $SU_q(1|2)$.

(i) $E^1 = 2q$

$$
\begin{aligned}
\Psi^1_1 &= |\uparrow\uparrow\uparrow\rangle \\
\Psi^1_2 &= q^2|\downarrow\uparrow\uparrow\rangle + q|\uparrow\downarrow\uparrow\rangle + |\uparrow\uparrow\downarrow\rangle \\
\Psi^1_3 &= q^2|0\uparrow\uparrow\rangle + q|\uparrow 0\uparrow\rangle + |\uparrow\uparrow 0\rangle \\
\Psi^1_4 &= q^2|\downarrow\downarrow\uparrow\rangle + q|\downarrow\uparrow\downarrow\rangle + |\uparrow\downarrow\downarrow\rangle \\
\Psi^1_5 &= q^3|0\downarrow\uparrow\rangle + q^2|0\uparrow\downarrow\rangle + q^2|\downarrow 0\uparrow\rangle + q|\downarrow\uparrow 0\rangle + q|\uparrow 0\downarrow\rangle + |\uparrow\downarrow 0\rangle \\
\Psi^1_6 &= q^2|0\downarrow\downarrow\rangle + q|\downarrow 0\downarrow\rangle + |\downarrow\downarrow 0\rangle \\
\Psi^1_7 &= |\downarrow\downarrow\downarrow\rangle
\end{aligned}
\tag{25}
$$

(ii) $E^2 = -2q^{-1}$

$$
\begin{aligned}
\Psi^2_1 &= |000\rangle \\
\Psi^2_2 &= |00\downarrow\rangle - q|0\downarrow 0\rangle + q^2|\downarrow 00\rangle \\
\Psi^2_3 &= |00\uparrow\rangle - q|0\uparrow 0\rangle + q^2|\uparrow 00\rangle \\
\Psi^2_4 &= |0\downarrow\uparrow\rangle - q|0\uparrow\downarrow\rangle - q|\downarrow 0\uparrow\rangle + q^2|\downarrow\uparrow 0\rangle + q^2|\uparrow 0\downarrow\rangle - q^3|\uparrow\downarrow 0\rangle
\end{aligned}
\tag{26}
$$

(iii) $E^3 = q - 1 - q^{-1}$

$$
\begin{aligned}
\Psi^3_1 &= |0\uparrow\downarrow\rangle - |\downarrow 0\uparrow\rangle + |\downarrow\uparrow 0\rangle - |\uparrow 0\downarrow\rangle \\
\Psi^3_2 &= -|\downarrow\downarrow\uparrow\rangle + q|\downarrow\uparrow\downarrow\rangle + |\downarrow\uparrow\downarrow\rangle - q|\uparrow\downarrow\downarrow\rangle \\
\Psi^3_3 &= -|\downarrow\uparrow\uparrow\rangle + q|\uparrow\downarrow\uparrow\rangle + |\uparrow\downarrow\uparrow\rangle - q|\uparrow\uparrow\downarrow\rangle \\
\Psi^3_4 &= -q|0\downarrow\uparrow\rangle + q|0\uparrow\downarrow\rangle + q^2|\downarrow 0\uparrow\rangle - |\downarrow 0\uparrow\rangle \\
&\quad -q^2|\downarrow\uparrow 0\rangle + q|\downarrow\uparrow 0\rangle + |\downarrow\uparrow 0\rangle - q|\uparrow\downarrow 0\rangle \\
\Psi^3_5 &= -|0\uparrow\uparrow\rangle + q|\uparrow 0\uparrow\rangle + |\uparrow 0\uparrow\rangle - q|\uparrow\uparrow 0\rangle \\
\Psi^3_6 &= -|0\downarrow\downarrow\rangle + q|\downarrow 0\downarrow\rangle + |\downarrow 0\downarrow\rangle - q|\downarrow\downarrow 0\rangle \\
\Psi^3_7 &= q|00\uparrow\rangle - q|0\uparrow 0\rangle + |0\uparrow 0\rangle - |\uparrow 00\rangle \\
\Psi^3_8 &= q|00\downarrow\rangle - q|0\downarrow 0\rangle + |0\downarrow 0\rangle - |\downarrow 00\rangle
\end{aligned}
\tag{27}
$$

(iv) $E^4 = q + 1 - q^{-1}$

$$\Psi_1^4 = |0\uparrow\uparrow\rangle - q|\uparrow 0\uparrow\rangle + |\uparrow 0\uparrow\rangle - q|\uparrow\uparrow 0\rangle$$
$$\Psi_2^4 = |0\downarrow\downarrow\rangle - q|\downarrow 0\downarrow\rangle + |\downarrow 0\downarrow\rangle - q|\downarrow\downarrow 0\rangle$$
$$\Psi_3^4 = q|00\uparrow\rangle + q|0\uparrow 0\rangle + |0\uparrow 0\rangle + |\uparrow 00\rangle$$
$$\Psi_4^4 = q|00\downarrow\rangle + q|0\downarrow 0\rangle + |0\downarrow 0\rangle + |\downarrow 00\rangle$$
$$\Psi_5^4 = -q|0\downarrow\uparrow\rangle + q^2|0\uparrow\downarrow\rangle - q|0\uparrow\downarrow\rangle - |0\uparrow\downarrow\rangle + q|\downarrow\uparrow 0\rangle + q^2|\uparrow 0\downarrow\rangle - |\uparrow 0\downarrow\rangle + q|\uparrow\downarrow 0\rangle \tag{28}$$
$$\Psi_6^4 = |\downarrow\uparrow\uparrow\rangle - q|\uparrow\downarrow\uparrow\rangle + |\uparrow\downarrow\uparrow\rangle - q|\uparrow\uparrow\downarrow\rangle$$
$$\Psi_7^4 = |\downarrow\downarrow\uparrow\rangle - q|\downarrow\uparrow\downarrow\rangle + |\downarrow\uparrow\downarrow\rangle - q|\uparrow\downarrow\downarrow\rangle$$
$$\Psi_8^4 = -|0\uparrow\downarrow\rangle + |\downarrow 0\uparrow\rangle + |\downarrow\uparrow 0\rangle - |\uparrow 0\downarrow\rangle .$$

From the above calculations of finite lattices it is obvious that the energy spectra vary with the parameter q. For $L = 3$ the energy gap between ground and first excited states is changed from 1 to $q + q^{-1} - 1$, and when q is taken to be negative, some energy levels are reversed.

For arbitrary lattice sites we consider the configuration

$$\Psi_\downarrow = \sum_{x=1}^{L} \alpha(x)|\uparrow\cdots\uparrow_{x-1}\downarrow_x\uparrow_{x+1}\cdots\uparrow\rangle . \tag{29}$$

The Schrödinger equation gives rise to

$$\begin{aligned} E\alpha(x) &= q(L-3)\alpha(x) + \alpha(x-1) + \alpha(x+1) + (q - q^{-1})\alpha(x) \qquad x \neq 1, L \\ E\alpha(1) &= q(L-2)\alpha(1) + (q - q^{-1})\alpha(1) + \alpha(2) \\ E\alpha(L) &= q(L-2)\alpha(L) + \alpha(L-1) . \end{aligned} \tag{30}$$

This equation set can be solved using the usual Bethe ansatz

$$\alpha(x) = A(k)e^{ikx} - A(-k)e^{-ikx} . \tag{31}$$

Substituting (31) into the first equation of (30) one can get eigenvalues of the Hamiltonian

$$E = (L-3)q + (q - q^{-1}) + 2\cos k . \tag{32}$$

Other equations in (30) give the ratio of the amplitude $A(k)$ and $A(-k)$

$$\frac{A(k)}{A(-k)} = \frac{1 - qe^{-ik}}{1 - qe^{ik}} \tag{33}$$

and

$$\frac{A(k)}{A(-k)} = \frac{(e^{-ik} - q - 1)e^{-ikL}}{(e^{ik} - q - 1)e^{ikL}} . \tag{34}$$

These two equations must be compatible, which gives a constraint on the impulse k,

$$e^{2Lk} = 1 . \tag{35}$$

Therefore

$$k = \frac{l\pi}{L} \qquad l = 0, 1, \ldots, 2L - 1 \tag{36}$$

and

$$A(k) = 1 - q\,e^{-ik} . \tag{37}$$

Now we analyse the spectrum in this special case. First, we note that $k = 0$ and $k = \pi$ should be ruled out since they give a vanishing Bethe ansatz function. Second, one can divide the possible impulses k into two parts, I: $(\pi/L, 2\pi/L, \ldots, (L-1)\pi/L)$ and II: $(\pi + \pi/L, \pi + 2\pi/L, \ldots, 2\pi - \pi/L)$. By redefining $\tilde{k}_j = 2\pi - k_j$, $k_j \in$ II, one can show that they give the same Bethe wavefunctions as $k_j \in I$. Therefore, we have only $L-1$ possible impulses and $L-1$ independent wavevectors. On the other hand, for the configuration with $L-1$ spin-up and one spin-down electrons, there exist L independent states, while the Bethe ansatz gives only $L-1$ states. The missing state can be compensated for by using the $SU_q(1|2)$ algebra operator $\mathcal{X}^{-+}$ acting on the all-spin-up state. This state has the same eigenenergy as the all-spin-up state because $\mathcal{X}^{-+}$ and H_{tj}^{q} are commutative.

From this simple special case, we see that the Bethe ansatz equation can give all of the energy spectrum, but not the complete states. This had been pointed out in other integrable models [8, 9]. It is worth noting that the Bethe ansatz states are the highest-weight states on which the symmetric group acting gives all complete states. For other configurations, this conclusion is also true.

For the configuration with one hole and $L-1$ spin-up electrons, we can similarly write down the wavefunction,

$$\Psi_0 = \sum_{x=1}^{L} \beta(x) |\uparrow \ldots \uparrow_{x-1} 0_x \uparrow_{x+1} \ldots \uparrow\rangle . \tag{38}$$

The Schrödinger equation gives rise to

$$\begin{aligned} E\beta(x) &= q(L-3)\beta(x) - \beta(x-1) - \beta(x+1) \qquad x \neq 1, L \\ E\beta(1) &= q(L-2)\beta(1) - \beta(2) + (q - q^{-1})\beta(1) \\ E\beta(L) &= q(L-2)\beta(L) - \beta(L-1) . \end{aligned} \tag{39}$$

Using Bethe ansatz it is easy to find

$$E = (L-3)q + (q - q^{-1}) + 2\cos k . \tag{40}$$

$$\beta(x) = (-1)^x \left\{ (1 - q\mathrm{e}^{-\mathrm{i}k})\mathrm{e}^{\mathrm{i}kx} - (1 - q\mathrm{e}^{\mathrm{i}k})\mathrm{e}^{-\mathrm{i}kx} \right\} \tag{41}$$

and

$$k = \frac{\pi}{L}, \frac{2\pi}{L}, \ldots, \frac{(L-1)\pi}{L} . \tag{42}$$

These give $L-1$ independent states. The missing state can be obtained by using the operator $\mathcal{X}^{0+}$ acting on the ferromagnetic states.

One can again see that the symmetry of the system helps in making up the missing states in Bethe ansatz approach. Hence investigating the symmetry of integrable system is not an artificial technique. It is the basis in studying the completeness of the Hilbert space of the system and discussing related physical properties. For detailed Bethe ansatz solutions of this generalized t–j model see [10].

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